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ORIGINAL PAPER

The p-step iterative algorithm for a system of generalized mixed quasi-variational inclusions with (H, η)-monotone operators

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Abstract In this paper, we introduce and study a new system of generalized mixed quasi-variational inclusions with (H, η)-monotone operators which contains variational inequalities, variational inclusions, systems of variational inequalities and systems of variational inclusions in the literature as special cases. By using the resolvent technique for the (H, η)-monotone operators, we prove the existence of solutions and the convergence of some new p-step iterative algorithms for this system of generalized mixed quasi-variational inclusions and its special cases. The results in this paper unifies, extends and improves some known results in the literature.

Keywords System of generalized mixed quasi-variational inclusions · (H, η)-monotone operator · Existence · p-step iterative algorithm · Convergence

1 Introduction

Variational inclusion problems are among the most interesting and intensively studied classes of mathematical problems and have wide applications in the fields of optimization and control, economics and transportation equilibrium, engineering science. For the past years, many existence results and iterative algorithms for various variational inequality and variational inclusion problems have been studied. For details, please see [1–47] and the references therein.

Recently, some new and interesting problems, which are called to be system of variational inequality problems were introduced and studied. Pang [27], Cohen and

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Chaplais [28], Bianchi [29] and Ansari and Yao [15] considered a system of scalar variational inequalities and Pang showed that the traffic equilibrium problem, the spatial equilibrium problem, the Nash equilibrium, and the general equilibrium programming problem can be modeled as a system of variational inequalities. Ansari et al. [30] introduced and studied a system of vector equilibrium problems and a system of vector variational inequalities by a fixed point theorem. Allevi et al. [31] considered a system of generalized vector variational inequalities and established some existence results with relative pseudomonotonicity. Kassay and Kolumbán [16] introduced a system of variational inequalities and proved an existence theorem by the Ky Fan lemma. Kassay et al. [17] studied Minty and Stampacchia variational inequality systems with the help of the Kakutani-Fan-Glicksberg fixed point theorem. Peng [18,19] introduced a system of quasi-variational inequality problems and proved its existence theorem by maximal element theorems. Verma [20–24] introduced and studied some systems of variational inequalities and developed some iterative algorithms for approximating the solutions of system of variational inequalities in Hilbert spaces. Kim and Kim [25] introduced a new system of generalized nonlinear quasi-variational inequalities and obtained some existence and uniqueness results of solution for this system of generalized nonlinear quasi-variational inequalities in Hilbert spaces. Cho et al. [26] introduced and studied a new system of nonlinear variational inequalities in Hilbert spaces. They proved some existence and uniqueness theorems of solutions for the system of nonlinear variational inequalities.

As generalizations of above systems of variational inequalities, Agarwal et al. [32] introduced a system of generalized nonlinear mixed quasi-variational inclusions and investigated the sensitivity analysis of solutions for this system of generalized nonlinear mixed quasi-variational inclusions in Hilbert spaces. Kazmi and Bhat [33] introduced a system of nonlinear variational-like inclusions and gave an iterative algorithm for finding its approximate solution. Fang and Huang [34], Fang et al. [35] introduced and studied a new system of variational inclusions involving H-monotone operators and (H,η)-monotone operators, respectively. Yan et al. [36] introduced and studied a system of set-valued variational inclusions which is more general than the model in [34]. Peng and Zhu [37] introduced and studied a new system of generalized mixed quasi-variational inclusions involving (H,η)-monotone operators which contains those mathematical models in [21–26,34–36] as special cases.

Inspired and motivated by the results in [15–37], the purpose of this paper is to introduce a new mathematical model, which is called to be a system of generalized mixed quasi-variational inclusions with (H,η)-monotone operators, i.e., a family of generalized mixed quasi-variational inclusions with (H,η)-monotone operators defined on a product set. This new mathematical model contains the system of inequalities in [15,20–29] and the system of inclusions in [34–37], the variational inclusions in [1, 2,11] and some variational inequalities in the literature as special cases. By using the resolvent technique for the (H,η)-monotone operators, we prove the existence of solutions for this system of set-valued mixed quasi-variational inclusions. We also prove the convergence of some p-step iterative algorithms approximating the solution for this system of generalized mixed quasi-variational inclusions and its special cases. The results in this paper unifies, extends and improves some results in [1,2,11,20–29, 34–37].

2 Preliminaries

We suppose that $\mathcal{H}$ is a real Hilbert space with norm and inner product denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let $CB(\mathcal{H})$ denote the families of all nonempty closed bounded subsets of $\mathcal{H}$, and $\tilde{D}(\cdot,\cdot)$ denote the Hausdorff metric on $CB(\mathcal{H})$ defined by

$$\tilde{D}(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\}, \quad \forall A,B \in CB(\mathcal{H}),$$

where $d(a,B) = \inf_{b\in B}\|a-b\|$, $d(A,b) = \inf_{a\in A}\|a-b\|$.

Now we recall some definitions needed later.

Definition 2.1 [35]. Let $\eta\colon \mathcal{H}\times\mathcal{H} \longrightarrow \mathcal{H}$ and $H : \mathcal{H} \longrightarrow \mathcal{H}$ be two single-valued operators and $M : \mathcal{H} \longrightarrow 2^{\mathcal{H}}$ be a set-valued operator. M is said to be

(i) η-monotone if,

$$\langle x-y, \eta(u,v)\rangle \geq 0, \forall u,v \in \mathcal{H}, x \in Mu, y \in Mv.$$

(ii) (H,η)-monotone if M is η-monotone and $(H+\lambda M)(\mathcal{H}) = \mathcal{H}$, for all $\lambda > 0$.

Remark 2.1 (1) It is easy to know that if $H = I$ (the identity map on $\mathcal{H}$), then the definition of (I,η)-monotone operators is that of maximal η-monotone operators.

(2) If $\eta(u,v) = u - v$, then the definition of η-monotonicity is that of monotonicity, the definition of (H,η)-monotonicity becomes that of H-monotonicity in [1] and the definition of (I,η)-monotone operators is that of maximal monotone operators.

(3) Hence, the class of (H,η)-monotone operators provides a unifying frameworks for classes of maximal monotone operators, maximal η-monotone operators, H-monotone operators. For more details about the above definitions, please refer [1,34–38] and the references therein.

Definition 2.2 [1,38] Let $H, g : \mathcal{H} \longrightarrow \mathcal{H}$, $\eta : \mathcal{H}\times\mathcal{H} \longrightarrow \mathcal{H}$ be three single-valued operators. g is said to be

(i) monotone if

$$\langle gu - gv, u - v\rangle \geq 0, \quad \forall u,v \in \mathcal{H};$$

(ii) strictly monotone if g is monotone and

$$\langle gu - gv, u - v\rangle = 0 \text{ if and only if } u = v;$$

(iii) strongly monotone if there exists a constant $r > 0$ such that

$$\langle gu - gv, u - v\rangle \geq r\|u - v\|^2, \quad \forall u,v \in \mathcal{H}.$$

(iv) Lipschitz continuous if there exists a constant $s > 0$ such that

$$\|g(u) - g(v)\| \leq s\|u - v\|, \quad \forall u,v \in \mathcal{H}.$$

(v) strongly monotone with respect to H if there exists a constant $\gamma > 0$ such that

$$\langle gu - gv, Hu - Hv\rangle \geq \gamma\|u - v\|^2, \quad u,v \in \mathcal{H}.$$

(vi) η-monotone if

$$\langle gu - gv, \eta(u,v)\rangle \geq 0, \quad \forall u,v \in \mathcal{H};$$

(vii) strictly η-monotone if g is η-monotone and

$$\langle gu - gv, \eta(u, v)\rangle = 0 \text{ if and only if } u = v;$$

(viii) strongly η-monotone if there exists a constant $r > 0$ such that

$$\langle gu - gv, \eta(u, v)\rangle \geq r\|u - v\|^2, \quad \forall u, v \in \mathcal{H}.$$

Definition 2.3 [38] Let $\eta : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$ be a single-valued operator, then $\eta(.,.)$ is said to be Lipschitz continuous, if there exists a constant $\tau > 0$ such that

$$\|\eta(u, v)\| \leq \tau\|u - v\|, \forall u, v \in \mathcal{H}.$$

Remark 2.2 Let $\mathcal{H} = R$ be the real number space, $\eta_1 : R \times R \longrightarrow R$, $\eta_2 : R \times R \longrightarrow R$ and $\eta_3 : R \times R \longrightarrow R$ be defined by $\eta_1(u, v) = u - v, \forall u, v \in R$, $\eta_2(u, v) = \frac{1}{2}(u - v), \forall u, v \in R$, $\eta_3(u, v) = 2\sin[\frac{1}{3}(u - v)], \forall u, v \in R$. It is easy to check that η_1, η_2 and η_3 are all Lipschitz continuous functions.

Definition 2.4 [35] Let $\eta : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$ be a single-valued operator, $H : \mathcal{H} \longrightarrow \mathcal{H}$ be a strongly η-monotone operator and $M : \mathcal{H} \longrightarrow 2^{\mathcal{H}}$ be an (H, η)-monotone operator. Then the resolvent operator $R^{H,\eta}_{M,\lambda} : \mathcal{H} \longrightarrow \mathcal{H}$ is defined by

$$R^{H,\eta}_{M,\lambda}(x) = (H + \lambda M)^{-1}(x), \quad \forall x \in \mathcal{H}.$$

Definition 2.5 [39] Let $M : \mathcal{H} \longrightarrow CB(\mathcal{H})$ be a set-valued mapping. Then M is said to be $\widetilde{D}$-Lipschitz continuous if there exists a constant $\xi > 0$ such that

$$\widetilde{D}(M(u), M(v)) \leq \xi\|u - v\|, \quad \forall u, v \in \mathcal{H}.$$

We also need the following result obtained by Fan et al. [35].

Lemma 2.1 *Let $\eta : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$ be a single-valued Lipschitz continuous operator with constant τ, $H : \mathcal{H} \longrightarrow \mathcal{H}$ be a strongly η-monotone operator with constant $\gamma > 0$ and $M : \mathcal{H} \longrightarrow 2^{\mathcal{H}}$ be an (H, η)-monotone operator. Then, the resolvent operator $R^{H,\eta}_{M,\lambda} : \mathcal{H} \longrightarrow \mathcal{H}$ is Lipschitz continuous with constant $\frac{\tau}{\gamma}$, i.e.,*

$$\|R^{H,\eta}_{M,\lambda}(x) - R^{H,\eta}_{M,\lambda}(y)\| \leq \frac{\tau}{\gamma}\|x - y\|, \quad \forall x, y \in \mathcal{H}.$$

We extend some definitions in [37, 39, 45] to more general cases as follows.

Definition 2.6 Let $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_p$ be Hilbert spaces, $g_1 : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$ and $N_1 : \prod_{j=1}^{p} \mathcal{H}_j \longrightarrow \mathcal{H}_1$ be two single-valued mappings.

(i) N_1 is said to be Lipschitz continuous in the first argument if there exists a constant $\xi > 0$ such that

$$\|N_1(x_1, x_2, \ldots, x_p) - N_1(y_1, x_2, \ldots, x_p)\| \leq \xi\|x_1 - y_1\|,$$
$$\forall x_1, y_1 \in \mathcal{H}_1, x_j \in \mathcal{H}_j \ \ (j = 2, 3, \ldots, p).$$

(ii) N_1 is said to be monotone in the first argument if

$$\langle N_1(x_1, x_2, \ldots, x_p) - N_1(y_1, x_2, \ldots, x_p), x_1 - y_1\rangle \geq 0,$$
$$\forall x_1, y_1 \in \mathcal{H}_1, x_j \in \mathcal{H}_j \ \ (j = 2, 3, \ldots, p).$$

(iii) N_1 is said to be strongly monotone in the first argument if there exists a constant $\alpha > 0$ such that

$$\langle N_1(x_1, x_2, \ldots, x_p) - N_1(y_1, x_2, \ldots, x_p), x_1 - y_1 \rangle \geq \alpha \|x_1 - y_1\|^2,$$
$$\forall x_1, y_1 \in \mathcal{H}_1, x_j \in \mathcal{H}_j \ \ (j = 2, 3, \ldots, p).$$

(iv) N_1 is said to be monotone with respect to g in the first argument if

$$\langle N_1(x_1, x_2, \ldots, x_p) - N_1(y_1, x_2, \ldots, x_p), g(x_1) - g(y_1) \rangle \geq 0,$$
$$\forall x_1, y_1 \in \mathcal{H}_1, x_j \in \mathcal{H}_j \ \ (j = 2, 3, \ldots, p).$$

(v) N_1 is said to be strongly monotone with respect to g in the first argument if there exists a constant $\beta > 0$ such that

$$\langle N_1(x_1, x_2, \ldots, x_p) - N_1(y_1, x_2, \ldots, x_p), g(x_1) - g(y_1) \rangle \geq \beta \|x_1 - y_1\|^2,$$
$$\forall x_1, y_1 \in \mathcal{H}_1, x_j \in \mathcal{H}_j \ \ (j = 2, 3, \ldots, p).$$

In a similar way, we can define the Lipschitz continuity and the strong monotonicity (monotonicity) of $N_i : \prod_{j=1}^{p} \mathcal{H}_j \longrightarrow \mathcal{H}_i$ (with respect to $g_i : \mathcal{H}_i \longrightarrow \mathcal{H}_i$) in the ith argument ($i = 2, 3, \ldots, p$).

3 A system of generalized mixed quasi-variational inclusions and a p-step iterative algorithm

In this section, we will introduce a new system of generalized mixed quasi-variational inclusions with (H, η)-monotone operators and construct some new p-step iterative algorithm for solving this system of generalized mixed quasi-variational inclusions and its special cases in Hilbert spaces. In what follows, unless other specified, for each $i = 1, 2, \ldots, p$, we always suppose that $\mathcal{H}_i$ is a Hilbert space, $H_i, g_i : \mathcal{H}_i \longrightarrow \mathcal{H}_i$, $\eta_i : \mathcal{H}_i \times \mathcal{H}_i \longrightarrow \mathcal{H}_i$, $F_i, G_i : \prod_{j=1}^{p} \mathcal{H}_j \longrightarrow \mathcal{H}_i$ are single-valued mappings, $T_{1i} : \mathcal{H}_1 \longrightarrow CB(\mathcal{H}_i)$, $T_{2i} : \mathcal{H}_2 \longrightarrow CB(\mathcal{H}_i)$,...,$T_{pi} : \mathcal{H}_p \longrightarrow CB(\mathcal{H}_i)$ are set-valued mappings and M_i: $\mathcal{H}_i \longrightarrow 2^{\mathcal{H}_i}$ is an (H_i, η_i)-monotone operator. We consider the following problem of finding $(x_1, x_2, \ldots, x_p, y_{11}, y_{12}, \ldots, y_{1p}, y_{21}, y_{22}, \ldots, y_{2p}, \ldots, y_{p1}, y_{p2}, \ldots, y_{pp})$ such that for each $i = 1, 2, \ldots, p$, $x_i \in \mathcal{H}_i$, $y_{1i} \in T_{1i}(x_1)$, $y_{2i} \in T_{2i}(x_2)$, ..., $y_{pi} \in T_{pi}(x_p)$ and

$$0 \in F_i(x_1, x_2, \ldots, x_p) + G_i(y_{i1}, y_{i2}, \ldots, y_{ip}) + M_i(g_i(x_i)). \tag{3.1}$$

The Problem (3.1) is called a system of generalized mixed quasi-variational inclusions with (H, η)-monotone operators.
Below are some special cases of Problem (3.1).

(i) If $p = 2$, then Problem (3.1) reduces to the following problem finding $(x_1, x_2, y_{11}, y_{12}, y_{21}, y_{22})$ such that $(x_1, x_2) \in \mathcal{H}_1 \times \mathcal{H}_2$, $y_{11} \in T_{11}(x_1)$, $y_{12} \in T_{12}(x_1)$, $y_{21} \in T_{21}(x_2)$, $y_{22} \in T_{22}(x_2)$ and

$$\begin{cases} 0 \in F_1(x_1, x_2) + G_1(y_{11}, y_{12}) + M_1(g_1(x_1)), \\ 0 \in F_2(x_1, x_2) + G_2(y_{21}, y_{22}) + M_2(g_2(x_2)). \end{cases} \tag{3.2}$$

Problem (3.2) was introduced and studied by Peng and Zhu [37].

(ii) If $p = 1$, then Problem (3.1) reduces to the following variational inclusions, which is to find (x_1, y_{11}) such that $x_1 \in \mathcal{H}_1$, $y_{11} \in T_{11}(x_1)$ and

$$0 \in F_1(x_1) + G_1(y_{11}) + M_1(g_1(x_1)). \tag{3.3}$$

If $T_{11} = H_1 \equiv I_1$ ((the identity map on $\mathcal{H}_1$)) and M be a maximal monotone operator, then Problem (3.3) becomes the variational inclusion introduced and studied by Adly [11] which contains the variational inequality in [2] as a special case.

If $G_1 \equiv 0$, $g_1 \equiv I_1$ and M is a H-monotone operator, then Problem (3.3) becomes the variational inclusion in [1] which contains the nonlinear variational inequality and the classical variational inequality (i.e., problem (3.2) and (3.3) in [1]) as special cases.

(iii) For each $j = 1, 2, \ldots, p$, if $g_j \equiv I_j$ (the identity map on $\mathcal{H}_j$) and $G_j \equiv 0$, then Problem (3.1) reduces to the system of variational inclusions with (H, η)-monotone operators, which is to find $(x_1, x_2, \ldots, x_p) \in \prod_{j=1}^{p} \mathcal{H}_j$ such that

$$0 \in F_i(x_1, x_2, \ldots, x_p) + M_i(x_i). \tag{3.4}$$

If $p = 2$, then Problem (3.4) becomes the system of variational inclusions with (H, η)-monotone operators in [35] which contains the system of variational inclusions with H-monotone operators in [34] as a special case.

(iv) For each $j = 1, 2, \ldots, p$, if $g_j \equiv I_j$, $\eta_j(x_j, y_j) = x_j - y_j$ for all $x_j, y_j \in \mathcal{H}_j$, and $F_j \equiv 0$, then Problem (3.1) reduces to the system of set-valued variational inclusions with H-monotone operators, which is to find $(x_1, x_2, \ldots, x_p, y_{11}, y_{12}, \ldots, y_{1p}, y_{21}, y_{22}, \ldots, y_{2p}, \ldots, y_{p1}, y_{p2}, \ldots, y_{pp})$ such that for each $i = 1, 2, \ldots, p$, $x_i \in \mathcal{H}_i$, $y_{1i} \in T_{1i}(x_1)$, $y_{2i} \in T_{2i}(x_2)$, ..., $y_{pi} \in T_{pi}(x_p)$ and

$$0 \in G_i(y_{i1}, y_{i2}, \ldots, y_{ip}) + M_i(x_i). \tag{3.5}$$

If $p = 2$, then Problem (3.5) becomes a system of set-valued variational inclusions with H-monotone operators which contains the mathematical model in [36] as a special case.

(v) For each $j = 1, 2, \ldots, p$, if $M_j(x_j) = \partial_{\eta_j}\varphi_j(x_j)$ for all $x_j \in \mathcal{H}_j$, where $\varphi_j : \mathcal{H}_j \longrightarrow R \cup \{+\infty\}$ is a proper, η_j-subdifferentiable functional and $\partial_{\eta_j}\varphi_j$ denotes the η_j-subdifferential operator of φ_j, then Problem (3.4) reduces to the following system of variational-like inequalities, which is to find $(x_1, x_2, \ldots, x_p) \in \prod_{i=1}^{p} \mathcal{H}_i$ such that for each $i = 1, 2, \ldots, p$,

$$\langle F_i(x_1, x_2, \ldots, x_p), \eta_i(z_i, x_i)\rangle + \varphi_i(z_i) - \varphi_i(x_i) \geq 0, \quad \forall z_i \in \mathcal{H}_i. \tag{3.6}$$

(vi) For each $j = 1, 2, \ldots, p$, if $M_j(x_j) = \partial\varphi_j(x_j)$, for all $x_j \in \mathcal{H}_j$, where $\varphi_j : \mathcal{H}_j \longrightarrow R \cup \{+\infty\}$ is a proper, convex, lower semicontinuous functional and $\partial\varphi_j$ denotes the subdifferential operator of φ_j, then Problem (3.4) reduces to the following system of variational inequalities, which is to find $(x_1, x_2, \ldots, x_p) \in \prod_{i=1}^{p} \mathcal{H}_i$ such that for each $i = 1, 2, \ldots, p$,

$$\langle F_i(x_1, x_2, \ldots, x_p), z_i - x_i\rangle + \varphi_i(z_i) - \varphi_i(x_i) \geq 0, \quad \forall z_i \in \mathcal{H}_i. \tag{3.7}$$

(vii) For each $j = 1, 2, \ldots, p$, if $M_j(x_j) = \partial\delta_{K_j}(x_j)$ for all $x_j \in \mathcal{H}_j$, where $K_j \subset \mathcal{H}_j$ is a nonempty, closed and convex subsets and δ_{K_j} denotes the indicator of K_j, then

Problem (3.7) reduces to the following system of variational inequalities, which is to find $(x_1, x_2, \ldots, x_p) \in \prod_{i=1}^{p} \mathcal{H}_i$ such that for each $i = 1, 2, \ldots, p$,

$$\langle F_i(x_1, x_2, \ldots, x_p), z_i - x_i \rangle \geq 0, \quad \forall z_i \in K_i. \tag{3.8}$$

Problem (3.8) was introduced and researched in [15,27–29]. If $p = 2$, then Problems (3.6), (3.7) and (3.8), respectively, become problems (3.2), (3.3) and (3.4) in [35]. It is easy to see that Problem (3.4) in [35] contains the models of system of variational inequalities in [20–24] as special cases.

It is worthy noting that Problem (3.1), problems (3.4)–(3.7) are all new problems.

Lemma 3.1 *For $i = 1, 2, \ldots, p$, let $\eta_i : \mathcal{H}_i \times \mathcal{H}_i \longrightarrow \mathcal{H}_i$ be a single-valued operator, $H_i : \mathcal{H}_i \longrightarrow \mathcal{H}_i$ be a strictly η_i-monotone operator and $M_i : \mathcal{H}_i \longrightarrow 2^{\mathcal{H}_i}$ is an (H_i, η_i)-monotone operator. Then $(x_1, x_2, \ldots, x_p, y_{11}, y_{12}, \ldots, y_{1p}, y_{21}, y_{22}, \ldots, y_{2p}, \ldots, y_{p1}, y_{p2}, \ldots, y_{pp})$ with $x_i \in \mathcal{H}_i$, $y_{1i} \in T_{1i}(x_1)$, $y_{2i} \in T_{2i}(x_2)$, ..., $y_{pi} \in T_{pi}(x_p)$ $(i = 1, 2, \ldots, p)$ is a solution of the Problem (3.1) if and only if for each $i = 1, 2, \ldots, p$,*

$$g_i(x_i) = R_{M_i,\lambda_i}^{H_i,\eta_i}(H_i(g_i(x_i)) - \lambda_i F_i(x_1, x_2, \ldots, x_p) - \lambda_i G_i(y_{i1}, y_{i2}, \ldots, y_{ip})),$$

where $R_{M_i,\lambda_i}^{H_i,\eta_i} = (H_i + \lambda_i M_i)^{-1}$, $\lambda_i > 0$ is a constant.

Proof The fact directly follows from Definition 2.4 □

For any given $x_i^0 \in \mathcal{H}_i$ $(i = 1, 2, \ldots, p)$, take $y_{1i}^0 \in T_{1i}(x_1^0)$, $y_{2i}^0 \in T_{2i}(x_2^0)$, ..., $y_{pi}^0 \in T_{pi}(x_p^0)$ $(i = 1, 2, \ldots, p)$. For $i = 1, 2, \ldots, p$, let

$$x_i^1 = x_i^0 - g_i\left(x_i^0\right) + R_{M_i,\lambda_i}^{H_i,\eta_i}\left(H_i\left(g_i\left(x_i^0\right)\right) - \lambda_i F_i\left(x_1^0, x_2^0, \ldots, x_p^0\right) - \lambda_i G_i\left(y_{i1}^0, y_{i2}^0, \ldots, y_{ip}^0\right)\right).$$

Since $y_{1i}^0 \in T_{1i}(x_1^0)$, $y_{2i}^0 \in T_{2i}(x_2^0)$, ..., $y_{pi}^0 \in T_{pi}(x_p^0)$ $(i = 1, 2, \ldots, p)$, by Nadler's Theorem [48], there exist $y_{1i}^1 \in T_{1i}(x_1^1)$, $y_{2i}^1 \in T_{2i}(x_2^1)$, ..., $y_{pi}^1 \in T_{pi}(x_p^1)$ $(i = 1, 2, \ldots, p)$, such that for each $i = 1, 2, \ldots, p$,

$$\begin{aligned}
\|y_{1i}^1 - y_{1i}^0\| &\leq (1+1)\tilde{D}\big(T_{1i}(x_1^1), T_{1i}(x_1^0)\big),\\
\|y_{2i}^1 - y_{2i}^0\| &\leq (1+1)\tilde{D}\big(T_{2i}(x_1^1), T_{2i}(x_1^0)\big),\\
&\cdots\\
\|y_{pi}^1 - y_{pi}^0\| &\leq (1+1)\tilde{D}\big(T_{pi}(x_1^1), T_{pi}(x_1^0)\big).
\end{aligned}$$

For $i = 1, 2, \ldots, p$, let

$$x_i^2 = x_i^1 - g_i\left(x_i^1\right) + R_{M_i,\lambda_i}^{H_i,\eta_i}\left(H_i\left(g_i\left(x_i^1\right)\right) - \lambda_i F_i\left(x_1^1, x_2^1, \ldots, x_p^1\right) - \lambda_i G_i\left(y_{i1}^1, y_{i2}^1, \ldots, y_{ip}^1\right)\right).$$

Again by Nadler's Theorem, there exist $y_{1i}^2 \in T_{1i}(x_1^2)$, $y_{2i}^2 \in T_{2i}(x_2^2)$, ..., $y_{pi}^2 \in T_{pi}(x_p^2)$ $(i = 1, 2, \ldots, p)$, such that for each $i = 1, 2, \ldots, p$,

$$\begin{aligned}
\|y_{1i}^2 - y_{1i}^1\| &\leq \left(1 + \frac{1}{2}\right)\tilde{D}(T_{1i}(x_1^2), T_{1i}(x_1^1)),\\
\|y_{2i}^2 - y_{2i}^1\| &\leq \left(1 + \frac{1}{2}\right)\tilde{D}(T_{2i}(x_1^2), T_{2i}(x_1^1)),\\
&\cdots\\
\|y_{pi}^2 - y_{pi}^1\| &\leq \left(1 + \frac{1}{2}\right)\tilde{D}(T_{pi}(x_1^2), T_{pi}(x_1^1)).
\end{aligned}$$

By induction, we can obtain the following p-step iterative algorithm for solving problem (3.1) as following:

Algorithm 3.1 For any given $x_i^0 \in \mathcal{H}_i$ $(i = 1, 2, \ldots, p)$, we can compute the sequences x_i^n, y_{1i}^n, y_{2i}^n, ..., y_{pi}^n $(i = 1, 2, \ldots, p)$ by the following p-step iterative schemes such that for each $i = 1, 2, \ldots, p$,

$$x_i^{n+1} = x_i^n - g_i(x_i^n) + R_{M_i,\lambda_i}^{H_i,\eta_i}\big(H_i(g_i(x_i^n)) - \lambda_i F_i(x_1^n, x_2^n, \ldots, x_p^n) - \lambda_i G_i(y_{i1}^n, y_{i2}^n, \ldots, y_{ip}^n)\big). \quad (3.9)$$

$$y_{1i}^n \in T_{1i}(x_1^n), \|y_{1i}^n - y_{1i}^{n-1}\| \le \left(1 + \frac{1}{n}\right)\tilde{D}\big(T_{1i}(x_1^n), T_{1i}(x_1^{n-1})\big), \quad (3.10)$$

$$y_{2i}^n \in T_{2i}(x_2^n), \|y_{2i}^n - y_{2i}^{n-1}\| \le \left(1 + \frac{1}{n}\right)\tilde{D}\big(T_{2i}(x_2^n), T_{2i}(x_2^{n-1})\big), \quad (3.11)$$

$$\cdots$$

$$y_{pi}^n \in T_{pi}(x_p^n), \|y_{pi}^n - y_{pi}^{n-1}\| \le \left(1 + \frac{1}{n}\right)\tilde{D}\big(T_{pi}(x_p^n), T_{pi}(x_p^{n-1})\big). \quad (3.12)$$

for all $n = 0, 1, 2, \ldots$.

we give a p-step iterative algorithm for solving Problem (3.4) as follows:

Algorithm 3.2 For any given $x_i^0 \in \mathcal{H}_i$ $(i = 1, 2, \ldots, p)$, we can compute the sequences x_i^n $(i = 1, 2, \ldots, p)$ by the following p-step iterative schemes such that for each $i = 1, 2, \ldots, p$,

$$x_i^{n+1} = R_{M_i,\lambda_i}^{H_i,\eta_i}\big(H_i(x_i^n) - \lambda_i F_i(x_1^n, x_2^n, \ldots, x_p^n)\big). \quad (3.13)$$

for all $n = 0, 1, 2, \ldots$.

Let $E = E^* = \mathcal{H}$ be a Hilbert space, we can rewrite Definition 2.3 introduced by Ahmad and Siddiqi [46] as follows.

Definition 3.1 Let $\eta: \mathcal{H} \times \mathcal{H}$ and $H: \mathcal{H} \to \mathcal{H}$ be two single-valued mappings, $\varphi: \mathcal{H} \to R \cup \{+\infty\}$ be a proper, η-subdifferentiable functional. If for each $x \in \mathcal{H}$ and for any $\rho > 0$, there exist a unique point $u \in \mathcal{H}$ satisfying

$$\langle Hu - x, \eta(y, u)\rangle \ge \rho\varphi(u) - \rho\varphi(y), \quad \forall y \in \mathcal{H},$$

then the mapping $x \mapsto u$, denoted by $J_{\partial_\eta\varphi}^{H,\lambda}$ is said to be J^η-proximal mapping of φ.

By the definition of $J_{\partial_\eta\varphi}^{H,\lambda}$, we have $x - Hu \in \rho\partial_\eta\varphi(u)$, it follows that

$$J_{\partial_\eta\varphi}^{H,\lambda}(x) = (H + \rho\partial_\eta\varphi)^{-1}(x).$$

Remark 3.1 If $H = I$, then the definition of J^η-proximal mapping of φ becomes that of the η-proximal mapping of φ in [47].

we also give a p-step iterative algorithm for solving problem (3.6) as follows:

Algorithm 3.3 For any given $x_i^0 \in \mathcal{H}_i$ $(i = 1, 2, \ldots, p)$, we can compute the sequences x_i^n $(i = 1, 2, \ldots, p)$ by the following p-step iterative schemes such that for each $i = 1, 2, \ldots, p$,

$$x_i^{n+1} = J_{\partial_{\eta_j}\varphi_j}^{H_i,\lambda_i}\left(x_i^n - \lambda_i F_i(x_1^n, x_2^n, \ldots, x_p^n)\right). \quad (3.14)$$

for all $n = 0, 1, 2, \ldots$, where $J_{\partial_{\eta_j}\varphi_j}^{H_i,\lambda_i} = (H_i + \lambda_i\partial_{\eta_j}\varphi_j)^{-1}$.

4 Existence of solutions and convergence of some iterative algorithms

We first prove the existence of solutions for Problem (3.1) and the convergence of the p-step iterative sequences generated by Algorithm 3.1.

Theorem 4.1 *For $i = 1,2,\ldots,p$, let $\eta_i : \mathcal{H}_i \times \mathcal{H}_i \longrightarrow \mathcal{H}_i$ be Lipshitz continuous with constant τ_i, $H_i : \mathcal{H}_i \longrightarrow \mathcal{H}_i$ be strongly η_i-monotone and Lipschitz continuous with constant γ_i and δ_i, respectively, $g_i : \mathcal{H}_i \longrightarrow \mathcal{H}_i$ be strongly monotone and Lipschitz continuous with constant r_i and s_i, respectively, $F_i : \prod_{k=1}^{p} \mathcal{H}_k \longrightarrow \mathcal{H}_i$ be strongly monotone with respect to $\hat{g}_i$ in the ith argument with constant $\alpha_i > 0$, Lipschitz continuous in the jth argument with constant $\beta_{ij} > 0$ for $j = 1,\ldots,i-1,i,i+1,\ldots,p$, where $\hat{g}_i : \mathcal{H}_i \longrightarrow \mathcal{H}_i$ is defined by $\hat{g}_i(x_i) = H_i \circ g_i(x_i) = H_i(g_i(x_i)), \forall x_i \in \mathcal{H}_i$, and $G_i : \prod_{k=1}^{p} \mathcal{H}_k \longrightarrow \mathcal{H}_i$ be Lipschitz continuous in the jth argument with constant $\xi_{ij} > 0$ for $j = 1,2,\ldots,p$, $M_i : \mathcal{H}_i \longrightarrow 2^{\mathcal{H}_i}$ be an (H_i,η_i)-monotone operator, and the set-valued mappings $T_{1i} : \mathcal{H}_1 \longrightarrow CB(\mathcal{H}_i)$, $T_{2i} : \mathcal{H}_2 \longrightarrow CB(\mathcal{H}_i)$,…,$T_{pi} : \mathcal{H}_p \longrightarrow CB(\mathcal{H}_i)$ be $\tilde{D}$-Lipschitz continuous with constants $l_{1i} > 0$, $l_{2i} > 0$,…, $l_{pi} > 0$, respectively. If there exist constants $\lambda_i > 0$ $(i = 1,2,\ldots,p)$ such that*

$$\begin{cases} \sqrt{1-2r_1+{s_1}^2} + \frac{\tau_1}{\gamma_1}\sqrt{{\delta_1}^2{s_1}^2 - 2\lambda_1\alpha_1 + \lambda_1^2{\beta_{11}}^2} \\ \quad + \frac{\tau_1}{\gamma_1}\lambda_1\left[\xi_{11}l_{11} + \sum_{j=2}^{p}(\beta_{j1} + \xi_{j1}l_{j1})\right] < 1, \\ \sqrt{1-2r_2+{s_2}^2} + \frac{\tau_2}{\gamma_2}\sqrt{{\delta_2}^2{s_2}^2 - 2\lambda_2\alpha_2 + \lambda_2^2{\beta_{22}}^2} \\ \quad + \frac{\tau_2}{\gamma_2}\lambda_2\left[(\beta_{12}+\xi_{12}l_{12}) + \xi_{22}l_{22} + \sum_{j=3}^{p}(\beta_{j2} + \xi_{j2}l_{j2})\right] < 1, \\ \ldots \\ \sqrt{1-2r_p+{s_p}^2} + \frac{\tau_p}{\gamma_p}\sqrt{{\delta_p}^2{s_p}^2 - 2\lambda_p\alpha_p + \lambda_p^2{\beta_{pp}}^2} \\ \quad + \frac{\tau_p}{\gamma_p}\lambda_p\left[\sum_{j=1}^{p-1}(\beta_{jp} + \xi_{jp}l_{jp}) + \xi_{pp}l_{pp}\right] < 1. \end{cases} \tag{4.1}$$

Then Problem (3.1) admits a solution $(x_1,x_2,\ldots,x_p,y_{11}, y_{12},\ldots,y_{1p},y_{21}, y_{22},\ldots,y_{2p},\ldots,y_{p1},y_{p2},\ldots,y_{pp})$ and sequences $x_1^n,x_2^n,\ldots,x_p^n,y_{11}^n,y_{12}^n,\ldots,y_{1p}^n,y_{21}^n,y_{21}^n$, $\ldots,y_{2p}^n,\ldots,y_{p1}^n,y_{p2}^n,\ldots,y_{pp}^n$ converge to $x_1,x_2,\ldots,x_p,y_{11},y_{12},\ldots,y_{1p},y_{21},y_{22},\ldots,y_{2p},\ldots$, $y_{p1},y_{p2},\ldots,y_{pp}$, respectively, where x_i^n, y_{1i}^n, y_{2i}^n, …, y_{pi}^n $(i = 1,2,\ldots,p)$ are the sequences generated by Algorithm 3.1.

Proof For $i = 1,2,\ldots,p$, let $\Omega_i^n \equiv H_i(g_i(x_i^n)) - \lambda_i F_i(x_1^n,x_2^n,\ldots,x_p^n) - \lambda_i G_i(y_{i1}^n,y_{i2}^n,\ldots,y_{ip}^n)$.

By (3.9) and Lemma 2.1, we have,

$$\begin{aligned} \left\|x_i^{n+1} - x_i^n\right\| &= \Big\|x_i^n - g_i(x_i^n) + R_{M_i,\lambda_i}^{H_i,\eta_i}(H_i(g_i(x_i^n)) - \lambda_i F_i(x_1^n,x_2^n,\ldots,x_p^n) - \lambda_i G_i(y_{i1}^n,y_{i2}^n,\ldots,y_{ip}^n)) \\ &\quad -\Big[x_i^{n-1} - g_i(x_i^{n-1}) + R_{M_i,\lambda_i}^{H_i,\eta_i}(H_i(g_i(x_i^{n-1})) \\ &\quad -\lambda_i F_i(x_1^{n-1},x_2^{n-1},\ldots,x_p^{n-1}) - \lambda_i G_i(y_{i1}^{n-1},y_{i2}^{n-1},\ldots,y_{ip}^{n-1}))\Big]\Big\| \\ &\le \left\|x_i^n - x_i^{n-1} - \left[g_i(x_i^n) - g_i(x_i^{n-1})\right]\right\| + \left\|R_{M_i,\lambda_i}^{H_i,\eta_i}(\Omega_i^n) - R_{M_i,\lambda_i}^{H_i,\eta_i}(\Omega_i^{n-1})\right\| \\ &\le \left\|x_i^n - x_i^{n-1} - \left[g_i(x_i^n) - g_i(x_i^{n-1})\right]\right\| + \frac{\tau_i}{\gamma_i}\left\|\Omega_i^n - \Omega_i^{n-1}\right\|, \quad i = 1,2,\ldots,p. \end{aligned} \tag{4.2}$$

Since $g_i: \mathcal{H}_i \longrightarrow \mathcal{H}_i$ is strongly monotone and Lipschitz continuous with constant r_i and s_i, respectively, we have,

$$\begin{aligned}
&\|x_i^n - x_i^{n-1} - [g_i(x_i^n) - g_i(x_i^{n-1})]\|^2 \\
&\quad \le \|x_i^n - x_i^{n-1}\|^2 - 2\langle g_i(x_i^n) - g_i(x_i^{n-1}), x_i^n - x_i^{n-1}\rangle + \|g_i(x_i^n) - g_i(x_i^{n-1})\|^2 \\
&\quad \le (1 - 2r_i + s_i{}^2)\|x_i^n - x_i^{n-1}\|^2, \quad i = 1, 2, \ldots, p. \qquad (4.3)
\end{aligned}$$

And

$$\begin{aligned}
&\|\Omega_i^n - \Omega_i^{n-1}\| \\
&\quad = \|H_i(g_i(x_i^n)) - \lambda_i F_i(x_1^n, x_2^n, \ldots, x_p^n) \\
&\qquad -\lambda_i G_i(y_{i1}^n, y_{i2}^n, \ldots, y_{ip}^n) \\
&\qquad -[H_i(g_i(x_i^{n-1})) - \lambda_i F_i(x_1^{n-1}, x_2^{n-1}, \ldots, x_p^{n-1}) \\
&\qquad -\lambda_i G_i(y_{i1}^{n-1}, y_{i2}^{n-1}, \ldots, y_{ip}^{n-1})]\| \\
&\quad \le \|H_i(g_i(x_i^n)) - H_i(g_i(x_i^{n-1})) - \lambda_i[F_i(x_1^n, \ldots, x_{i-1}^n, x_i^n, x_{i+1}^n \ldots, x_p^n) \\
&\qquad -F_i(x_1^n, \ldots, x_{i-1}^n, x_i^{n-1}, x_{i+1}^n \ldots, x_p^n)]\| \\
&\qquad +\lambda_i\|F_i(x_1^n, \ldots, x_{i-1}^n, x_i^{n-1}, x_{i+1}^n \ldots, x_p^n) \\
&\qquad -F_i(x_1^{n-1}, \ldots, x_{i-1}^{n-1}, x_i^{n-1}, x_{i+1}^{n-1} \ldots, x_p^{n-1})\| \\
&\qquad +\lambda_i\|G_i(y_{i1}^n, y_{i2}^n, \ldots, y_{ip}^n) - G_i(y_{i1}^{n-1}, y_{i2}^{n-1}, \ldots, y_{ip}^{n-1})\|, \quad i = 1, 2, \ldots, p. \qquad (4.4)
\end{aligned}$$

Since $F_i: \prod_{k=1}^p \mathcal{H}_k \longrightarrow \mathcal{H}_i$ is strongly monotone with respect to $\hat{g}_i$ in the ith argument with constant $\alpha_i > 0$, and Lipschitz continuous in the ith argument with constant $\beta_{ii} > 0$, respectively, we get,

$$\begin{aligned}
&\|H_i(g_i(x_i^n)) - H_i(g_i(x_i^{n-1})) - \lambda_i[F_i(x_1^n, \ldots, x_{i-1}^n, x_i^n, x_{i+1}^n \ldots, x_p^n) \\
&\qquad -F_i(x_1^n, \ldots, x_{i-1}^n, x_i^{n-1}, x_{i+1}^n \ldots, x_p^n)]\|^2 \\
&\quad \le \|H_i(g_i(x_i^n)) - H_i(g_i(x_i^{n-1}))\|^2 \\
&\qquad -2\lambda_i\langle F_i(x_1^n, \ldots, x_{i-1}^n, x_i^n, x_{i+1}^n \ldots, x_p^n) \\
&\qquad -F_i(x_1^n, \ldots, x_{i-1}^n, x_i^{n-1}, x_{i+1}^n \ldots, x_p^n), H_i(g_i(x_i^n)) - H_i(g_i(x_i^{n-1}))\rangle \\
&\qquad +\lambda_i^2\|F_i(x_1^n, \ldots, x_{i-1}^n, x_i^n, x_{i+1}^n \ldots, x_p^n) \\
&\qquad -F_i(x_1^n, \ldots, x_{i-1}^n, x_i^{n-1}, x_{i+1}^n \ldots, x_p^n)\|^2 \\
&\quad \le (\delta_i{}^2 s_i{}^2 - 2\lambda_i\alpha_i + \lambda_i^2\beta_{ii}{}^2)\|x_i^n - x_i^{n-1})\|^2, \quad i = 1, 2, \ldots p. \qquad (4.5)
\end{aligned}$$

Since $F_i: \prod_{k=1}^p \mathcal{H}_k \longrightarrow \mathcal{H}_i$ is Lipschitz continuous in the jth argument with constant $\beta_{ij} > 0$ for $j = 1, \ldots, i-1, i+1, \ldots, p$, we have,

$$
\begin{aligned}
&\|F_i(x_1^n,\ldots,x_{i-1}^n,x_i^{n-1},x_{i+1}^n\ldots,x_p^n)-F_i(x_1^{n-1},\ldots,x_{i-1}^{n-1},x_i^{n-1},x_{i+1}^{n-1}\ldots,x_p^{n-1})\|\\
&\quad\le \|F_i(x_1^n,x_2^n,\ldots,x_{i-1}^n,x_i^{n-1},x_{i+1}^n\ldots,x_p^n)\\
&\qquad -F_i(x_1^{n-1},x_2^n,\ldots,x_{i-1}^n,x_i^{n-1},x_{i+1}^n\ldots,x_p^n)\|\\
&\qquad +\|F_i(x_1^{n-1},x_2^n,x_3^n,\ldots,x_{i-1}^n,x_i^{n-1},x_{i+1}^n\ldots,x_p^n)\\
&\qquad -F_i(x_1^{n-1},x_2^{n-1},x_3^n,\ldots,x_{i-1}^n,x_i^{n-1},x_{i+1}^n\ldots,x_p^n)\|\\
&\qquad +\|F_i(x_1^{n-1},x_2^{n-1},x_3^{n-1},\ldots,x_{i-1}^n,x_i^{n-1},x_{i+1}^n\ldots,x_p^n)\\
&\qquad -F_i(x_1^{n-1},x_2^{n-1},x_3^{n-1},\ldots,x_{i-1}^{n-1},x_i^{n-1},x_{i+1}^n\ldots,x_p^n)\|\\
&\qquad +\|F_i(x_1^{n-1},x_2^{n-1},x_3^{n-1},\ldots,x_{i-1}^{n-1},x_i^{n-1},x_{i+1}^n\ldots,x_p^n)\\
&\qquad -F_i(x_1^{n-1},x_2^{n-1},x_3^{n-1},\ldots,x_{i-1}^{n-1},x_i^{n-1},x_{i+1}^{n-1}\ldots,x_p^n)\|\\
&\qquad +\|F_i(x_1^{n-1},x_2^{n-1},\ldots,x_{i-1}^{n-1},x_i^{n-1},x_{i+1}^{n-1}\ldots,x_p^n)\\
&\qquad -F_i(x_1^{n-1},x_2^{n-1},\ldots,x_{i-1}^{n-1},x_i^{n-1},x_{i+1}^{n-1}\ldots,x_p^{n-1})\|\\
&\quad\le \beta_{i1}\|x_1^n-x_1^{n-1}\|+\beta_{i2}\|x_2^n-x_2^{n-1}\|+\ldots+\beta_{i,i-1}\|x_{i-1}^n-x_{i-1}^{n-1}\|\\
&\qquad +\beta_{i,i+1}\|x_{i-1}^n-x_{i+1}^{n-1}\|+\ldots\\
&\quad +\beta_{i,p}\|x_p^n-x_p^{n-1}\|=\sum_{j=1}^{i-1}\beta_{ij}\|x_j^n-x_j^{n-1}\|\\
&\qquad +\sum_{j=i+1}^{p}\beta_{ij}\|x_j^n-x_j^{n-1}\|,\quad i=1,2,\ldots p. \qquad (4.6)
\end{aligned}
$$

It follows from the Lipschitz continuity of G_i, the $\tilde{D}$-Lipschitz continuity of T_{ij}, (3.10), (3.11) and (3.12) that

$$
\begin{aligned}
&\left\|G_i(y_{i1}^n,y_{i2}^n,\ldots,y_{ip}^n)-G_i(y_{i1}^{n-1},y_{i2}^{n-1},\ldots,y_{ip}^{n-1})\right\|\\
&\quad\le \left\|G_i(y_{i1}^n,y_{i2}^n,\ldots,y_{ip}^n)-G_i(y_{i1}^{n-1},y_{i2}^n,\ldots,y_{ip}^n)\right\|\\
&\qquad+\left\|G_i(y_{i1}^{n-1},y_{i2}^n,\ldots,y_{ip}^n)-G_i(y_{i1}^{n-1},y_{i2}^{n-1},\ldots,y_{ip}^n)\right\|\\
&\qquad+\left\|G_i(y_{i1}^{n-1},y_{i2}^{n-1},\ldots,y_{ip}^n)-G_i(y_{i1}^{n-1},y_{i2}^{n-1},\ldots,y_{ip}^{n-1})\right\|\\
&\quad\le \sum_{j=1}^{p}\xi_{ij}\left\|y_{ij}^n-y_{ij}^{n-1}\right\|\le\sum_{j=1}^{p}\xi_{ij}\left(1+\frac{1}{n}\right)\tilde{D}(T_{ij}(x_i^n),T_{ij}(x_i^{n-1}))\\
&\quad\le \sum_{j=1}^{p}\xi_{ij}\left(1+\frac{1}{n}\right)l_{ij}\left\|x_i^n-x_i^{n-1}\right\|,\quad i=1,2,\ldots,n. \qquad (4.7)
\end{aligned}
$$

It follows from (4.2)–(4.7) that for $i=1,2,\ldots,p$,

$$
\begin{aligned}
\|x_i^{n+1}-x_i^n\| &\le \Bigg[\sqrt{1-2r_i+s_i{}^2}\\
&\quad+\frac{\tau_i}{\gamma_i}\left(\sqrt{\delta_i{}^2 s_i{}^2-2\lambda_i\alpha_i+\lambda_i^2\beta_{ii}{}^2}+\lambda_i\xi_{ii}l_{ii}\left(1+\frac{1}{n}\right)\right)\Bigg]\left\|x_i^n-x_i^{n-1}\right\|
\end{aligned}
$$

$$+\sum_{j=1}^{i-1}\lambda_j\left(\beta_{ij}+\xi_{ij}l_{ij}\left(1+\frac{1}{n}\right)\right)\frac{\tau_j}{\gamma_j}\left\|x_j^n-x_j^{n-1}\right\|$$
$$+\sum_{j=i+1}^{p}\lambda_j\left(\beta_{ij}+\xi_{ij}l_{ij}\left(1+\frac{1}{n}\right)\right)\frac{\tau_j}{\gamma_j}\left\|x_j^n-x_j^{n-1}\right\|. \tag{4.8}$$

Therefore,

$$\begin{aligned}
\sum_{i=1}^{p}\|x_i^{n+1}-x_i^n\| &\le \sum_{i=1}^{p}\Bigg\{\Bigg[\sqrt{1-2r_i+s_i{}^2}\\
&\quad+\frac{\tau_i}{\gamma_i}\left(\sqrt{\delta_i{}^2s_i{}^2-2\lambda_i\alpha_i+\lambda_i^2\beta_{ii}{}^2}+\lambda_i\xi_{ii}l_{ii}\left(1+\frac{1}{n}\right)\right)\Bigg]\|x_i^n-x_i^{n-1}\|\\
&\quad+\sum_{j=1}^{i-1}\lambda_j\left(\beta_{ij}+\xi_{ij}l_{ij}\left(1+\frac{1}{n}\right)\right)\frac{\tau_j}{\gamma_j}\|x_j^n-x_j^{n-1}\|\\
&\quad+\sum_{j=i+1}^{p}\lambda_j\left(\beta_{ij}+\xi_{ij}l_{ij}\left(1+\frac{1}{n}\right)\right)\frac{\tau_j}{\gamma_j}\|x_j^n-x_j^{n-1}\|\Bigg\}\\
&=\Bigg(\sqrt{1-2r_1+s_1{}^2}+\frac{\tau_1}{\gamma_1}\sqrt{\delta_1{}^2s_1{}^2-2\lambda_1\alpha_1+\lambda_1^2\beta_{11}{}^2}\\
&\quad+\frac{\tau_1}{\gamma_1}\lambda_1\left[\xi_{11}l_{11}\left(1+\frac{1}{n}\right)+\sum_{j=2}^{p}\left(\beta_{j1}+\xi_{j1}l_{j1}\left(1+\frac{1}{n}\right)\right)\right]\Bigg)\|x_1^n-x_1^{n-1}\|\\
&\quad+\Bigg(\sqrt{1-2r_2+s_2{}^2}+\frac{\tau_2}{\gamma_2}\sqrt{\delta_2{}^2s_2{}^2-2\lambda_2\alpha_2+\lambda_2^2\beta_{22}{}^2}+\frac{\tau_2}{\gamma_2}\lambda_2\Bigg[\left(\beta_{12}+\xi_{12}l_{12}\left(1+\frac{1}{n}\right)\right)\\
&\quad+\xi_{22}l_{22}\left(1+\frac{1}{n}\right)+\sum_{j=3}^{p}\left(\beta_{j2}+\xi_{j2}l_{j2}\left(1+\frac{1}{n}\right)\right)\Bigg]\Bigg)\|x_2^n-x_2^{n-1}\|+\ldots\\
&\quad+\Bigg(\sqrt{1-2r_p+s_p{}^2}+\frac{\tau_p}{\gamma_p}\sqrt{\delta_p{}^2s_p{}^2-2\lambda_p\alpha_p+\lambda_p^2\beta_{pp}{}^2}\\
&\quad+\frac{\tau_p}{\gamma_p}\lambda_p\left[\sum_{j=1}^{p-1}\left(\beta_{jp}+\xi_{jp}l_{jp}\left(1+\frac{1}{n}\right)\right)+\xi_{pp}l_{pp}\left(1+\frac{1}{n}\right)\right]\Bigg)\|x_p^n-x_p^{n-1}\|\\
&\le\theta_n\left(\sum_{i=1}^{p}\|x_i^n-x_i^{n-1}\|\right),
\end{aligned} \tag{4.9}$$

where

$$\begin{aligned}
\theta_n=\max\Bigg\{&\sqrt{1-2r_1+s_1{}^2}+\frac{\tau_1}{\gamma_1}\sqrt{\delta_1{}^2s_1{}^2-2\lambda_1\alpha_1+\lambda_1^2\beta_{11}{}^2}\\
&+\frac{\tau_1}{\gamma_1}\lambda_1\left[\xi_{11}l_{11}\left(1+\frac{1}{n}\right)+\sum_{j=2}^{p}\left(\beta_{j1}+\xi_{j1}l_{j1}\left(1+\frac{1}{n}\right)\right)\right],\\
&\sqrt{1-2r_2+s_2{}^2}+\frac{\tau_2}{\gamma_2}\sqrt{\delta_2{}^2s_2{}^2-2\lambda_2\alpha_2+\lambda_2^2\beta_{22}{}^2}\\
&+\frac{\tau_2}{\gamma_2}\lambda_2\left[\left(\beta_{12}+\xi_{12}l_{12}\left(1+\frac{1}{n}\right)\right)+\xi_{22}l_{22}\left(1+\frac{1}{n}\right)+\sum_{j=3}^{p}\left(\beta_{j2}+\xi_{j2}l_{j2}\left(1+\frac{1}{n}\right)\right)\right],
\end{aligned}$$

$$\ldots, \sqrt{1-2r_p+{s_p}^2}+\frac{\tau_p}{\gamma_p}\sqrt{{\delta_p}^2{s_p}^2-2\lambda_p\alpha_p+\lambda_p^2{\beta_{pp}}^2}$$
$$+\frac{\tau_p}{\gamma_p}\lambda_p\left[\sum_{j=1}^{p-1}\left(\beta_{jp}+\xi_{jp}l_{jp}\left(1+\frac{1}{n}\right)\right)+\xi_{pp}l_{pp}\left(1+\frac{1}{n}\right)\right]\Bigg\}.$$

Let

$$\theta=\max\left\{\sqrt{1-2r_1+{s_1}^2}+\frac{\tau_1}{\gamma_1}\sqrt{{\delta_1}^2{s_1}^2-2\lambda_1\alpha_1+\lambda_1^2{\beta_{11}}^2}+\frac{\tau_1}{\gamma_1}\lambda_1\left[\xi_{11}l_{11}+\sum_{j=2}^{p}\left(\beta_{j1}+\xi_{j1}l_{j1}\right)\right],\right.$$
$$\sqrt{1-2r_2+{s_2}^2}+\frac{\tau_2}{\gamma_2}\sqrt{{\delta_2}^2{s_2}^2-2\lambda_2\alpha_2+\lambda_2^2{\beta_{22}}^2}+\frac{\tau_2}{\gamma_2}\lambda_2\left[(\beta_{12}+\xi_{12}l_{12})+\xi_{22}l_{22}+\sum_{j=3}^{p}(\beta_{j2}+\xi_{j2}l_{j2})\right],$$
$$\left.\ldots,\sqrt{1-2r_p+{s_p}^2}+\frac{\tau_p}{\gamma_p}\sqrt{{\delta_p}^2{s_p}^2-2\lambda_p\alpha_p+\lambda_p^2{\beta_{pp}}^2}+\frac{\tau_p}{\gamma_p}\lambda_p\left[\sum_{j=1}^{p-1}(\beta_{jp}+\xi_{jp}l_{jp})+\xi_{pp}l_{pp}\right]\right\}.$$

Then $\theta_n \longrightarrow \theta$ as $n \longrightarrow \infty$. By (4.1), we know that $0<\theta<1$ and so (4.9) implies that $x_1^n, x_2^n, \ldots, x_p^n$ are all Cauchy sequences. Thus, there exist $x_1 \in \mathcal{H}_1$, $x_2 \in \mathcal{H}_2, \ldots$, $x_p \in \mathcal{H}_p$ such that $x_1^n \longrightarrow x_1, x_2^n \longrightarrow x_2, \ldots, x_p^n \longrightarrow x_p$ as $n \longrightarrow \infty$.

Now we prove that $y_{1i}^n \longrightarrow y_{1i} \in T_{1i}(x_1)$, $y_{2i}^n \longrightarrow y_{2i} \in T_{2i}(x_2)$, ..., $y_{pi}^n \longrightarrow y_{pi} \in T_{pi}(x_p)$ $(i=1,2,\ldots,p)$. In fact, it follows from the Lipschitz continuity of T_{1i}, T_{2i}, ..., T_{pi} and (3.14)–(3.22) that for $i=1,2,\ldots,p$,

$$\|y_{1i}^n-y_{1i}^{n-1}\| \le \left(1+\frac{1}{n}\right)l_{1i}\|x_1^n-x_1^{n-1}\|, \tag{4.10}$$
$$\|y_{2i}^n-y_{2i}^{n-1}\| \le \left(1+\frac{1}{n}\right)l_{2i}\|x_2^n-x_2^{n-1}\|, \tag{4.11}$$
$$\ldots$$
$$\|y_{pi}^n-y_{pi}^{n-1}\| \le \left(1+\frac{1}{n}\right)l_{pi}\|x_p^n-x_p^{n-1}\|. \tag{4.12}$$

From (4.10)–(4.12), we know that y_{1i}^n, y_{2i}^n, ..., y_{pi}^n($i=1,2,\ldots,p$) are also Cauchy sequences. Therefore, there exist $y_{1i} \in \mathcal{H}_1$, $y_{2i} \in \mathcal{H}_2$, ...,$y_{pi} \in \mathcal{H}_p$ such that $y_{1i}^n \longrightarrow y_{1i}$, $y_{2i}^n \longrightarrow y_{2i}$, ..., $y_{pi}^n \longrightarrow y_{pi}$ as $n \longrightarrow \infty$. Further, for $i=1,2,\ldots,p$,

$$\begin{aligned} d(y_{1i},T_{1i}(x_1)) &\le \left\|y_{1i}-y_{1i}^n\right\|+d(y_{1i}^n,(T_{1i}(x_1)))\\ &\le \left\|y_{1i}-y_{1i}^n\right\|+\tilde{D}((T_{1i}(x_1^n)),(T_{1i}(x_1)))\\ &\le \left\|y_{1i}-y_{1i}^n\right\|+l_{1i}\left\|x_1^n-x_1\right\| \longrightarrow 0. \end{aligned}$$

Since $T_{1i}(x_1)$ is closed, we have $y_{1i} \in T_{1i}(x_1)$ $(i=1,2,\ldots,p)$. Similarly, $y_{2i} \in T_{2i}(x_2),\ldots,y_{pi} \in T_{pi}(x_p)$ $(i=1,2,\ldots,p)$. By continuity of $g_i, H_i, F_i, G_i, T_{1i}, T_{2i}, \ldots, T_{pi}, R_{M_i,\lambda_i}^{H_i,\eta_i}$ and Algorithm 3.1, we know that $x_1,x_2,\ldots,x_p,y_{11},y_{12},\ldots,y_{1p},y_{21},y_{22},\ldots, y_{2p},\ldots,y_{p1},y_{p2},\ldots,y_{pp}$ satisfy the following relation,

$$g_i(x_i)=R_{M_i,\lambda_i}^{H_i,\eta_i}(H_i(g_i(x_i))-\lambda_iF_i(x_1,x_2,\ldots,x_p)-\lambda_iG_i(y_{i1},y_{i2},\ldots,y_{ip})), \quad i=1,2,\ldots,p$$

By Lemma 3.1, $(x_1,x_2,\ldots,x_p,y_{11},y_{12},\ldots,y_{1p},y_{21},y_{22},\ldots,y_{2p},\ldots,y_{p1},y_{p2},\ldots,y_{pp})$ is a solution of problem (3.1). This completes the proof. □

Remark 4.1 By the results in Sects. 3 and 4, it is easy to obtain the convergence results of iterative algorithms for the other special cases of Problem (3.1), now we give two examples as follows.

For $i = 1, 2, \ldots, p$, let $G_i = 0$, $g_i = I_i$, then $r_i = s_i = 1$, $\xi_{ij} = 0$ for $i, j = 1, 2, \ldots, p$. By Theorem 4.1, we have the following result.

Corollary 4.1 *For $i = 1, 2, \ldots, p$, let $\eta_i : \mathcal{H}_i \times \mathcal{H}_i \longrightarrow \mathcal{H}_i$ be Lipshitz continuous with constant τ_i, $H_i : \mathcal{H}_i \longrightarrow \mathcal{H}_i$ be strongly η_i-monotone and Lipschitz continuous with constant γ_i and δ_i, respectively, $F_i : \prod_{k=1}^{p} \mathcal{H}_k \longrightarrow \mathcal{H}_i$ be strongly monotone with respect to H_i in the ith argument with constant $\alpha_i > 0$, Lipschitz continuous in the jth argument with constant $\beta_{ij} > 0$ for $j = 1, \ldots, i-1, i, i+1, \ldots, p$, $M_i : \mathcal{H}_i \longrightarrow 2^{\mathcal{H}_i}$ be an (H_i, η_i)-monotone operator. If there exist constants $\lambda_i > 0$ $(i = 1, 2, \ldots, p)$ such that*

$$\begin{cases} \frac{\tau_1}{\gamma_1}\sqrt{{\delta_1}^2 - 2\lambda_1\alpha_1 + \lambda_1^2{\beta_{11}}^2} + \frac{\tau_1}{\gamma_1}\lambda_1\left(\sum\limits_{j=2}^{p}\beta_{j1}\right) < 1, \\ \frac{\tau_2}{\gamma_2}\sqrt{{\delta_2}^2 - 2\lambda_2\alpha_2 + \lambda_2^2{\beta_{22}}^2} + \frac{\tau_2}{\gamma_2}\lambda_2\left(\beta_{12} + \sum\limits_{j=3}^{p}\beta_{j2}\right) < 1, \\ \cdots \\ \frac{\tau_p}{\gamma_p}\sqrt{{\delta_p}^2 - 2\lambda_p\alpha_p + \lambda_p^2{\beta_{pp}}^2} + \frac{\tau_p}{\gamma_p}\lambda_p\left(\sum\limits_{j=1}^{p-1}\beta_{jp}\right) < 1. \end{cases} \tag{4.13}$$

Then Problem (3.4) admits a solution $(x_1, x_2, \ldots, x_p)$ and sequences $x_1^n, x_2^n, \ldots, x_p^n$ converge to $x_1, x_2, \ldots, x_p$ are the sequences generated by Algorithm 3.2.

Lemma 4.1 *Let $H : \mathcal{H} \to \mathcal{H}$ be strongly η-monotone with constant $\gamma > 0$, $\eta : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be Lipschitz continuous with constant $\tau > 0$ such that $\eta(x, y) = -\eta(y, x)$ for all $x, y \in \mathcal{H}$ and for any given $x \in \mathcal{H}$, the function $h(y, u) = \langle x - Hu, \eta(y, u)\rangle$ is 0-diagonally quasi-concave in y, $\varphi : \mathcal{H} \to R \cup \{+\infty\}$ be a proper, η-subdifferentiable functional. Then $\partial_\eta\varphi$ is (H, η)-monotone.*

Proof We can prove that $\partial_\eta\varphi$ is η-monotone. In fact, for any $x_1, x_2 \in \mathcal{H}$, $f_1^* \in \partial_\eta\varphi(x_1), f_2^* \in \partial_\eta\varphi(x_2)$, we have

$$\varphi(y) - \varphi(x_1) \geq \langle f_1^*, \eta(y, x_1)\rangle, \quad \forall y \in \mathcal{H}. \tag{4.14}$$

$$\varphi(y) - \varphi(x_2) \geq \langle f_2^*, \eta(y, x_2)\rangle, \quad \forall y \in \mathcal{H}. \tag{4.15}$$

Taking $y = x_2$ in (4.14) and $y = x_1$ in (4.15), and adding these inequalities, we obtain

$$\langle f_1^*, \eta(x_2, x_1)\rangle + \langle f_2^*, \eta(x_1, x_2)\rangle \leq 0. \tag{4.16}$$

Since $\eta(x_2, x_1) = -\eta(x_1, x_2)$ for all $x_1, x_2 \in \mathcal{H}$, we have

$$\langle f_1^* - f_2^*, \eta(x_1, x_2)\rangle \geq 0.$$

The above inequality implies that $\partial_\eta\varphi$ is η-monotone.

It follows from Theorem 2.1 in [46] that for any $\rho > 0$, and any $x \in \mathcal{H}$, there exists a unique $u \in \mathcal{H}$ such that $x - Hu \in \rho\partial_\eta\varphi(u)$. That is, $x \in (H + \rho\partial_\eta\varphi)(u)$. This implies that $\mathcal{H} \subseteq (H + \rho\partial_\eta\varphi)(\mathcal{H})$. And so $(H + \rho\partial_\eta\varphi)(\mathcal{H}) = \mathcal{H}$. This completes the proof. □

For $i = 1, 2, \ldots, p$, let $M_i = \partial_{\eta_i}\varphi_i$. By Corollary 4.1 and Lemma 4.1, we have the following result.

Corollary 4.2 *For $i = 1,2,\ldots,p$, let $H_i\colon \mathcal{H}_i \longrightarrow \mathcal{H}_i$ be strongly η_i-monotone and Lipschitz continuous with constant γ_i and δ_i, respectively, η_i: $\mathcal{H}_i \times \mathcal{H}_i \longrightarrow \mathcal{H}_i$ be Lipschitz continuous with constant τ_i such that $\eta_i(x_i,y_i) = -\eta_i(y_i,x_i)$ for all $x_i,y_i \in \mathcal{H}_i$ and for any given $x_i \in \mathcal{H}_i$, the function $h_i(y_i,u_i) = \langle x_i - H_iu_i, \eta_i(y_i,u_i)\rangle$ is 0-diagonally quasi-concave in y_i, $\varphi_j\colon \mathcal{H}_j \longrightarrow R \cup \{+\infty\}$ is a proper, η_j-subdifferentiable functional, $F_i\colon \prod_{k=1}^{p} \mathcal{H}_k \longrightarrow \mathcal{H}_i$ be strongly monotone with respect to H_i in the ith argument with constant $\alpha_i > 0$, Lipschitz continuous in the jth argument with constant $\beta_{ij} > 0$ for $j = 1,\ldots,i-1,i,i+1,\ldots,p$. If there exist constants $\lambda_i > 0$ $(i = 1,2,\ldots,p)$ such that*

$$\begin{cases} \frac{\tau_1}{\gamma_1}\sqrt{{\delta_1}^2 - 2\lambda_1\alpha_1 + \lambda_1^2{\beta_{11}}^2} + \frac{\tau_1}{\gamma_1}\lambda_1\left(\sum\limits_{j=2}^{p}\beta_{j1}\right) < 1, \\ \frac{\tau_2}{\gamma_2}\sqrt{{\delta_2}^2 - 2\lambda_2\alpha_2 + \lambda_2^2{\beta_{22}}^2} + \frac{\tau_2}{\gamma_2}\lambda_2\left(\beta_{12} + \sum\limits_{j=3}^{p}\beta_{j2}\right) < 1, \\ \cdots \\ \frac{\tau_p}{\gamma_p}\sqrt{{\delta_p}^2 - 2\lambda_p\alpha_p + \lambda_p^2{\beta_{pp}}^2} + \frac{\tau_p}{\gamma_p}\lambda_p\left(\sum\limits_{j=1}^{p-1}\beta_{jp}\right) < 1. \end{cases} \tag{4.17}$$

Then Problem (3.6) admits a solution $(x_1,x_2,\ldots,x_p)$ and sequences $x_1^n,x_2^n,\ldots,x_p^n$ converge to $x_1,x_2,\ldots,x_p$ are the sequences generated by Algorithm 3.3.

Remark 4.2 Theorem 4.1, Corollary 4.1 and Corollary 4.2 unifies, improves and extends those corresponding results in [20–26,34–37] in several aspects.

Remark 4.3 Problem (3.6) can not be treated in the classical framework of maximal monotone operators.

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